

# Basis of symmetric polynomials for many-boson light-front wave functions

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(Dated: September 24, 2014)

## Abstract

We provide an algorithm for the construction of orthonormal multivariate polynomials that are symmetric with respect to the interchange of any two coordinates on the unit hypercube and are constrained to the hyperplane where the sum of the coordinates is one. These polynomials form a basis for the expansion of bosonic light-front momentum-space wave functions, as functions of longitudinal momentum, where momentum conservation guarantees that the fractions are on the interval  $[0, 1]$  and sum to one. This generalizes earlier work on three-boson wave functions to wave functions for arbitrarily many identical bosons. A simple application in two-dimensional  $\phi^4$  theory illustrates the use of these polynomials.

PACS numbers: 02.60.Nm, 11.15.Tk, 11.10.Ef

## I. INTRODUCTION

To solve a quantum field theory nonperturbatively, numerical techniques are usually required. The most commonly used technique is lattice gauge theory [1]; however, this approach is Euclidean and lacks direct contact with wave functions. Without wave functions in a Minkowski metric, some physical observables can be difficult if not impossible to calculate. Access to wave functions provides for a much more direct approach. The technique of Dyson-Schwinger equations [2] rectifies this situation somewhat, but remains Euclidean and requires models for higher vertex functions. An alternative that can provide wave functions in Minkowski (momentum) space is the light-front Hamiltonian approach [3–5].

In light-front quantization, the state of a system is found as an eigenstate of the Hamiltonians  $\mathcal{P}^- \equiv \mathcal{P}^0 - \mathcal{P}^z$ ,  $\mathcal{P}^+ \equiv \mathcal{P}^0 + \mathcal{P}^z$ , and  $\vec{\mathcal{P}}_\perp \equiv (\mathcal{P}^x, \mathcal{P}^y)$ . Here  $\mathcal{P}^0$  is the equal-time Hamiltonian operator, and  $\vec{\mathcal{P}} = (\mathcal{P}^x, \mathcal{P}^y, \mathcal{P}^z)$  is the equal-time momentum operator. The light-front Hamiltonian  $\mathcal{P}^-$  evolves a system in light-front time  $x^+ \equiv t + z$ ; the momentum operator  $\mathcal{P}^+$  translates a system in  $x^- \equiv t - z$ . These are the light-front coordinates of Dirac [6]. The stationary eigenstates of  $\mathcal{P}^-$  can be expanded in a Fock basis consisting of eigenstates of  $\mathcal{P}^+$  and  $\vec{\mathcal{P}}_\perp$  and of the particle-number operator; only  $\mathcal{P}^-$  contains terms that change constituents and mix different Fock states. The coefficients of the Fock-state expansion are the light-front momentum-space wave functions.

In the longitudinal (plus) direction, the light-front momentum of the  $i$ th constituent  $p_i^+ = E_i + p_i^z$  is always positive. We can then define longitudinal momentum fractions  $x_i = p_i^+ / P^+$ , relative to the total momentum  $P^+$ . The wave functions are boost-invariant functions of these momentum fractions.<sup>1</sup> Momentum conservation requires that these fractions be on the interval  $[0, 1]$  and that the fractions sum to one.

In order that the Fock expansion be an eigenstate of  $\mathcal{P}^-$ , the wave functions must satisfy a system of integral equations. One way to solve such a system is to expand the wave functions in a truncated set of basis functions and solve the resulting matrix eigenvalue problem for the coefficients of the basis functions.

If a Fock state consists of identical bosons, the wave function must be symmetric in its arguments. If such a wave function is to be expanded in a basis, the basis functions should also be symmetric. If the momentum values are unconstrained, this is relatively straightforward, but here the longitudinal momenta *are* constrained. Therefore, for the dependence on longitudinal momenta, we need a set of basis functions that satisfy the symmetry requirement and the constraint. For the case of two bosons, this is quite simple, because there is only one independent variable. For three bosons, the analysis is somewhat complex; details can be found elsewhere [7]. Here we study the general case, wave functions for an arbitrary number of identical bosons.

Because the momentum fractions are limited to the interval  $[0, 1]$  and constrained to sum to one, the wave function for  $N$  bosons is defined on an  $(N - 1)$ -dimensional hyperplane within an  $N$ -dimensional hypercube. Thus, multivariate polynomials with the correct symmetry, combined with weight functions that control endpoint behavior, can be a convenient choice for basis functions.

The construction of multivariate symmetric polynomials on the  $N$ -dimensional hypercube is straightforward. The difficulty comes from the fact that, when restricted to the hyperplane

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<sup>1</sup> This boost invariance is one of the advantages of light-front quantization [3]. Another is that the Fock-state expansion itself is well defined; the positivity of the longitudinal momentum prevents spurious vacuum contributions.

defined by  $\sum_i^N x_i = 1$ , most of the polynomials of a given order are no longer linearly independent. For example, consider the case of  $N = 4$ . As is true for any  $N$ , there is no linear polynomial, because the only symmetric form on the hypercube,  $\sum_i^N x_i$ , is identically equal to one on the hyperplane. For second order polynomials, the two choices on the hypercube,  $\sum_i^4 x_i^2$  and  $\sum_{i \leq j}^4 x_i x_j$ , differ only by multiplicative and additive constants on the hyperplane. Similarly, the three possible third-order polynomials on the hypercube are linearly related on the hyperplane. At order four, however, we find two linearly independent polynomials on the hyperplane, reduced from five on the hypercube. These can be chosen to be the square of the second-order polynomial and the polynomial  $\prod_i^4 x_i$ ; any other symmetric polynomial of order four can be written as a linear combination of these two, plus lower-order polynomials.

Although it is possible, in principle, to continue working out polynomials order by order for fixed  $N$ , as in the  $N = 4$  example above, this is not very practical. Instead we have developed an algorithm by which the structure of these polynomials is known for arbitrary order and for an arbitrary number  $N$  of bosons. Orthonormality, with respect to any appropriate weight function, is then constructed order by order and polynomial by polynomial, with use of the Gramm-Schmidt process, which is easily automated.

The algorithm is derived and described in Sec. II. A simple application to two-dimensional  $\phi^4$  theory, to illustrate the use of these basis functions, is discussed in Sec. III; this includes a comparison with results obtained with the frequently used method of discrete light-cone quantization (DLCQ) [8]. A short summary is given in Sec. IV.

## II. ALGORITHM

We wish to construct linearly independent polynomials in  $N$  variables  $x_i$  that are fully symmetric on the unit hypercube and restricted to the hyperplane defined by  $\sum_i^N x_i = 1$ . Those symmetric on the hypercube are easily constructed, as

$$\tilde{P}_n^{(N)} = x_1^{\tilde{n}_1} x_2^{\tilde{n}_2} \cdots x_N^{\tilde{n}_N} + \text{permutations}, \quad (2.1)$$

with  $n = \tilde{n}_1 + \tilde{n}_2 + \cdots + \tilde{n}_N$  the order of the polynomial. To make the polynomial unique, the individual powers are restricted by the inequalities

$$\tilde{n}_1 \leq \tilde{n}_2 \leq \cdots \leq \tilde{n}_N. \quad (2.2)$$

Unfortunately, these polynomials are not linearly independent on the hyperplane  $\sum_i^N x_i = 1$ , as discussed in the Introduction.

Consider, however, a different construction. Given a set of  $N$  polynomials  $C_m$  on the hypercube, where  $C_m$  is of order  $m$ , a polynomial of order  $n$  can be built as

$$P_n^{(N)} = C_1^{n_1} C_2^{n_2} \cdots C_N^{n_N}, \quad (2.3)$$

with  $n = n_1 + 2n_2 + 3n_3 + \cdots + Nn_N$ . These polynomials have the distinct advantage that the restriction to the hyperplane is obvious; because  $C_1 = \sum_i x_i$  becomes 1, the linearly independent set is obtained by considering only those  $P_n^{(N)}$  for which  $n_1 = 0$ , provided the other  $C_m$  satisfy some restrictions, discussed below.

It is not immediately obvious that this new set is the same size as the first. Because it is linearly independent, it can be no larger, but it could be smaller. To see what happens,

we can simply count polynomials in each basis on the hypercube. This generalizes the proof for  $N = 3$ , given in the Appendix of [7].

For the first set, the  $\tilde{P}_n^{(N)}$ , we have the following number of polynomials of order  $n$ :

$$\tilde{S}_n^{(N)} = \sum_{\tilde{n}_1=0}^{\lfloor n/N \rfloor} \sum_{\tilde{n}_2=\tilde{n}_1}^{\lfloor (n-\tilde{n}_1)/(N-1) \rfloor} \cdots \sum_{\tilde{n}_i=\tilde{n}_{i-1}}^{\lfloor (n-\sum_{i'=1}^{i-1} \tilde{n}_{i'})/(N-i+1) \rfloor} \cdots \sum_{\tilde{n}_{N-1}=\tilde{n}_{N-2}}^{\lfloor (n-\sum_{i'=1}^{N-2} \tilde{n}_{i'})/2 \rfloor} 1, \quad (2.4)$$

with  $\lfloor x \rfloor$  being the integer part of  $x$ . The value of  $\tilde{n}_N$  is fixed at  $n - \sum_{i=1}^{N-1} \tilde{n}_i$ . The lower limits on the sums are determined by the constraint (2.2). The upper limit on  $\tilde{n}_1$  can be no higher than  $\lfloor n/N \rfloor$ , because the other  $N - 1$  indices must start at this upper limit and all together they must sum to  $n$ ; this would be impossible if  $\tilde{n}_1$  went beyond  $n/N$ . Similarly, for  $\tilde{n}_2$ , the upper limit must be the available total of  $n - \tilde{n}_1$  divided among the remaining  $N - 1$  indices. Continuing in this fashion, we determine all the upper bounds on the sums.<sup>2</sup>

For the second set of polynomials, the number of order  $n$  on the hypercube is

$$S_n^{(N)} = \sum_{n_N=0}^{\lfloor n/N \rfloor} \sum_{n_{N-1}=0}^{\lfloor (n-Nn_N)/(N-1) \rfloor} \cdots \sum_{n_i=0}^{\lfloor (n-\sum_{i'=i+1}^N i' n_{i'})/i \rfloor} \cdots \sum_{n_2=0}^{\lfloor (n-\sum_{i'=3}^N i' n_{i'})/2 \rfloor} 1, \quad (2.5)$$

with  $n_1 = n - \sum_{i=2}^N i n_i$ . The upper bounds are determined by the portion of the total order  $n$  that can be assigned to a particular polynomial  $C_m$ . In general, this is at most  $\lfloor n/m \rfloor$ . However, if other  $C_m$  factors have already been assigned some contribution to the total order, this contribution must first be subtracted from  $n$  before the division by  $m$ ; the upper bound on a particular sum takes this into account by subtracting from  $n$  the appropriate contribution already made in the sums to the left.

To show that the two counts  $\tilde{S}_n^{(N)}$  and  $S_n^{(N)}$  are the same, introduce to  $S_n^{(N)}$  the following change in summation indices:

$$n_i = \begin{cases} \tilde{n}_{N-i+1} - \tilde{n}_{N-i}, & i < N \\ \tilde{n}_1, & i = N. \end{cases} \quad (2.6)$$

The sum over  $n_i$  becomes

$$\sum_{n_i=0}^{\lfloor (n-\sum_{i'=i+1}^N i' n_{i'})/i \rfloor} = \sum_{\tilde{n}_{N-i+1}=\tilde{n}_{N-i}}^{\lfloor (n-\sum_{i'=i+1}^N i' (\tilde{n}_{N-i'+1} - \tilde{n}_{N-i'}) )/i \rfloor + \tilde{n}_{N-i}}}, \quad (2.7)$$

with the understanding that  $\tilde{n}_0 = 0$ . The upper limit can be simplified by taking advantage of cancellations. For example, the last two terms in the sum are  $(N-1)(\tilde{n}_2 - \tilde{n}_1) + N\tilde{n}_1 = \tilde{n}_1 + (N-1)\tilde{n}_2$ . The result is

$$\lfloor (n - \sum_{i'=i+1}^N i' (\tilde{n}_{N-i'+1} - \tilde{n}_{N-i'}) )/i \rfloor + \tilde{n}_{N-i} = \lfloor (n - \sum_{i'=1}^{N-i} \tilde{n}_{i'})/i - \tilde{n}_{N-i} \rfloor + \tilde{n}_{N-i} = \lfloor (n - \sum_{i'=1}^{N-i} \tilde{n}_{i'})/i \rfloor. \quad (2.8)$$

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<sup>2</sup> Notice that there is a typographical error in Eq. (A2) of [7]; the upper limit of the first sum should be  $\lfloor N/3 \rfloor$ .

As a final step, we use this reduction and replace  $i$  by  $N - i + 1$ , to obtain

$$\sum_{n_i=0}^{[(n-\sum_{i'=i+1}^N i' n_{i'})/i]} \rightarrow \sum_{\tilde{n}_i=\tilde{n}_{i-1}}^{[n-\sum_{i'=1}^{i-1} \tilde{n}_{i')/(N-i+1)]} . \quad (2.9)$$

Given this reduction of the individual sums, the count  $S_n^{(N)}$  takes the same form as  $\tilde{S}_n^{(N)}$ . Thus, the number of linearly independent polynomials on the hypercube is the same for the two forms  $P_n^{(N)}$  and  $\tilde{P}_n^{(N)}$ . The restriction to the hyperplane then selects the subset of the  $P_n^{(N)}$  with  $n_1 = 0$ .

The structure of symmetric polynomials in  $N$  variables on the hyperplane can now be written as linear combinations of the factorizations

$$P_{ni}^{(N)} = C_2^{n_2} C_3^{n_3} \cdots C_N^{n_N}, \quad (2.10)$$

with the indices restricted by  $n = \sum_j j n_j$ . The second subscript on  $P_{ni}^{(N)}$  differentiates between different linearly independent polynomials of the same order  $n$ . For  $n \geq 2$  there is always at least one such polynomial. For  $n \geq 4$ , there can be more than one, depending on the dimension  $N$  of the hypercube.

One caveat is that the factorization of  $P_{ni}^{(N)}$  assumes that none of the  $C_m$  can be written as a product of lower-order polynomials. Such products are already included directly in the factorization. For example, a choice of  $C_4 = C_2^2$  would mean that the seventh-order polynomial  $C_3 C_4$  is the same as the polynomial  $C_2^2 C_3$ . This must be avoided by a proper choice of the set  $\{C_m\}$ ; otherwise, the counting argument is not valid, because it assumes linear independence of the different factorizations.

A choice of the  $C_m$  that maintains the linear independence, though probably not unique, is to always write  $C_m$  as a product of the lowest order monomials available. They take the form of the  $\tilde{P}_n^{(N)}$ , as given in (2.1), with all indices  $\tilde{n}_i$  equal to zero or one and, of course, summing to  $n$ . For example,  $C_N$  would be  $x_1 x_2 \cdots x_N$  and  $C_{N-1}$  would be  $x_2 x_3 \cdots x_N + \text{permutations}$ . When restricted to the hyperplane,  $x_N$  is replaced by  $1 - \sum_{i=1}^{N-1} x_i$ ; this generates terms that are no more than quadratic in any individual  $x_i$ , not only for  $C_2$  but for all the  $C_m$ . Any product of the  $C_m$  will contain higher powers of  $x_i$  and will therefore be automatically linearly independent of any individual  $C_m$ .

All that remains to complete the algorithm is to specify the orthonormalization. This is done by the standard Gram-Schmidt construction, relative to a chosen inner product. Given a positive weight function  $w(x_i)$  on the hyperplane, the orthonormal combinations  $O_{ni}^{(N)}$  of the basis polynomials  $P_{ni}^{(N)}$  are chosen to satisfy

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-\sum_{j=1}^{N-2} x_j} dx_{N-1} w(x_i) O_{n'i'}^{(N)}(x_i) O_{ni}^{(N)}(x_i) = \delta_{n'n} \delta_{i'i}. \quad (2.11)$$

We would then naturally choose basis functions for  $N$ -boson wave functions as

$$f_{ni}^{(N)}(x_i) = \sqrt{w(x_i)} O_{ni}^{(N)}(x_i). \quad (2.12)$$

The choice of weight function is driven by the particular application and can be used to incorporate endpoint behavior in the basis functions. For  $N = 2$  and a weight function of the form  $x_1^\alpha x_2^\beta$ , the orthonormal polynomials generated by the given algorithm are proportional to the even-order Jacobi polynomials  $P_n^{(\alpha, \beta)}$ , transformed from  $[-1, 1]$  to  $[0, 1]$ .

### III. APPLICATION

As an example of how these polynomials can be used, we consider two-dimensional  $\phi^4$  theory. The light-front Hamiltonian is

$$\mathcal{P}^- = \mathcal{P}_{11}^- + \mathcal{P}_{22}^- + \mathcal{P}_{13}^- + \mathcal{P}_{31}^-, \quad (3.1)$$

with

$$\mathcal{P}_{11}^- = \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+) a(p^+), \quad (3.2)$$

$$\begin{aligned} \mathcal{P}_{22}^- = \frac{\lambda}{4} \int \frac{dp_1^+ dp_2^+}{4\pi \sqrt{p_1^+ p_2^+}} \int \frac{dp_1'^+ dp_2'^+}{\sqrt{p_1'^+ p_2'^+}} \delta(p_1^+ + p_2^+ - p_1'^+ - p_2'^+) \\ \times a^\dagger(p_1^+) a^\dagger(p_2^+) a(p_1'^+) a(p_2'^+), \end{aligned} \quad (3.3)$$

$$\mathcal{P}_{13}^- = \frac{\lambda}{6} \int \frac{dp_1^+ dp_2^+ dp_3^+}{4\pi \sqrt{p_1^+ p_2^+ p_3^+} (p_1^+ + p_2^+ + p_3^+)} a^\dagger(p_1^+ + p_2^+ + p_3^+) a(p_1^+) a(p_2^+) a(p_3^+), \quad (3.4)$$

$$\mathcal{P}_{31}^- = \frac{\lambda}{6} \int \frac{dp_1^+ dp_2^+ dp_3^+}{4\pi \sqrt{p_1^+ p_2^+ p_3^+} (p_1^+ + p_2^+ + p_3^+)} a^\dagger(p_1^+) a^\dagger(p_2^+) a^\dagger(p_3^+) a(p_1^+ + p_2^+ + p_3^+). \quad (3.5)$$

The boson creation and annihilation operators satisfy the commutation relation

$$[a(p^+), a^\dagger(p'^+)] = \delta(p^+ - p'^+). \quad (3.6)$$

The creation operators can be used to construct Fock states

$$|x_i P^+; P^+, n\rangle = \frac{1}{\sqrt{n!}} \prod_{i=1}^n a^\dagger(x_i P^+) |0\rangle \quad (3.7)$$

from the Fock vacuum  $|0\rangle$ . An eigenstate of  $\mathcal{P}^-$  can be written as an expansion in these Fock states

$$|\psi(P^+)\rangle = \sum_n (P^+)^{(n-1)/2} \int \left( \prod_{i=1}^{n-1} dx_i \right) \psi_n(x_1, \dots, x_n) |x_i P^+; P^+, n\rangle, \quad (3.8)$$

with  $\psi_n$  the  $n$ -boson wave function.

The terms of the Hamiltonian change particle number by zero or two, and, therefore, eigenstates can be classified as having an even or odd number of constituents. In [7] we considered the odd case, in order to have a Fock state with three bosons. Here, to have a more general application, we consider the even case and truncate the basis to include only two and four-boson Fock states. To simplify the analysis, we also truncate the Hamiltonian to discard the two-two scattering term  $\mathcal{P}_{22}^-$  in the four-boson sector; this allows the system of equations for the wave functions to be reduced to a single two-body equation, where a high-resolution calculation can be easily made, for the purpose of comparison.

The light-front eigenvalue problem [3]

$$\mathcal{P}^- |\psi(P^+)\rangle = \frac{M^2}{P^+} |\psi(P^+)\rangle \quad \text{and} \quad \mathcal{P}^+ |\psi(P^+)\rangle = P^+ |\psi(P^+)\rangle \quad (3.9)$$

yields the following coupled integral equations for the two-boson and four-boson wave functions:

$$M^2\psi_2 = \left(\frac{\mu^2}{x_1} + \frac{\mu^2}{x_2}\right)\psi_2 + \frac{1}{2}\frac{\lambda}{4\pi}\frac{1}{\sqrt{x_1x_2}}\int_0^1\frac{dx'_1}{\sqrt{x'_1x'_2}}\psi_2(x'_1, x'_2) \quad (3.10)$$

$$+ \frac{1}{\sqrt{3}}\frac{\lambda}{4\pi}\int_0^{x_1}dx'_1\int_0^{x_1-x'_1}dx'_2\left(\frac{\psi_4(x'_1, x'_2, x_1-x'_1-x'_2, x_2)}{\sqrt{x_1x'_1x'_2(x_1-x'_1-x'_2)}} + (x_1 \leftrightarrow x_2)\right),$$

$$M^2\psi_4 = \sum_{i=1}^4\frac{\mu^2}{x_i}\psi_4 + \frac{1}{2\sqrt{3}}\frac{\lambda}{4\pi}\left(\frac{\psi_2(x_1+x_2+x_3, x_4)}{\sqrt{x_1x_2x_3(x_1+x_2+x_3)}} + (x_1 \leftrightarrow x_4) + (x_2 \leftrightarrow x_4) + (x_3 \leftrightarrow x_4)\right). \quad (3.11)$$

The second equation can be solved explicitly for  $\psi_4$ . Substitution into the first equation provides a single, two-boson equation,

$$M^2\psi_2 = \left(\frac{\mu^2}{x_1} + \frac{\mu^2}{x_2}\right)\psi_2 + \frac{1}{2}\frac{\lambda}{4\pi}\frac{1}{\sqrt{x_1x_2}}\int_0^1\frac{dx'_1}{\sqrt{x'_1x'_2}}\psi_2(x'_1, x'_2) \quad (3.12)$$

$$+ \frac{1}{6}\left(\frac{\lambda}{4\pi}\right)^2\int_0^{x_1}\frac{dx'_1}{x'_1}\int_0^{x_1-x'_1}\frac{dx'_2}{x'_2}\left[\left(\frac{1}{M^2 - \frac{\mu^2}{x'_1} - \frac{\mu^2}{x'_2} - \frac{\mu^2}{x_1-x'_1-x'_2} - \frac{\mu^2}{x_2}}\right)\right.$$

$$\times \left(\frac{\psi_2(x_1, x_2)}{x_1(x_1-x'_1-x'_2)} + 3\frac{\psi_2(x_1-x'_1-x'_2, x'_1+x'_2+x_2)}{\sqrt{x_1x_2(x_1-x'_1-x'_2)(x'_1+x'_2+x_2)}} + (x_1 \leftrightarrow x_2)\right)\Big].$$

In each of these equations it is to be understood that the second momentum fraction for a two-boson system and the fourth momentum fraction for a four-boson system is not truly independent; the sum of momentum fractions in a wave function must be one.

To use the symmetric orthonormal polynomials  $O_{ni}^{(N)}$  to solve the system of equations, we approximate the wave functions by truncated sums

$$\psi_2(x_1, x_2) = \sqrt{x_1x_2}\sum_n^K a_n^{(2)}O_n^{(2)}(x_1) \quad (3.13)$$

and

$$\psi_4(x_1, x_2, x_3, x_4) = \sqrt{x_1x_2x_3x_4}\sum_{ni}^K a_{ni}^{(4)}O_{ni}^{(4)}(x_1, x_2, x_3). \quad (3.14)$$

The truncation at  $n = K$  is a truncation of the range of polynomial orders to a maximum of  $K$ . Although the truncations can be tuned separately in the different Fock sectors, to optimize a calculation, we do not do that here. Also, so that the  $N = 2$  polynomials change as the truncation  $K$  is relaxed, we take  $K$  to be even and increment in steps of 2 when studying convergence, there being no odd-order symmetric polynomials for two bosons.

On substitution of the polynomial expansions, the coupled integral equations (3.10) and (3.11) become

$$\frac{M^2}{\mu^2}a_m^{(2)} = A_{mn}^{(2)}a_n^{(2)} + \frac{1}{2}\frac{\lambda}{4\pi\mu^2}B_mB_na_n^{(2)} + \frac{2}{\sqrt{3}}\frac{\lambda}{4\pi\mu^2}C_{m,ni}a_{ni}^{(4)}, \quad (3.15)$$

$$\frac{M^2}{\mu^2}a_{mj}^{(4)} = A_{mj,ni}^{(4)}a_{ni}^{(4)} + \frac{2}{\sqrt{3}}\frac{\lambda}{4\pi\mu^2}C_{n,mj}a_n^{(2)}, \quad (3.16)$$

where repeated indices are summed and the matrices are

$$A_{mn}^{(2)} = 2 \int_0^1 dx_1 (1 - x_1) O_m^{(2)}(x_1) O_n^{(2)}(x_1), \quad (3.17)$$

$$A_{mj,ni}^{(4)} = 4 \int_0^1 dx_1 x_2 x_3 (1 - x_1 - x_2 - x_3) O_{mj}^{(4)}(x_1, x_2, x_3) O_{ni}^{(4)}(x_1, x_2, x_3), \quad (3.18)$$

$$B_n = \int_0^1 dx_1 O_n^{(2)}(x_1), \quad (3.19)$$

$$C_{m,ni} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 (1 - x_1 - x_2 - x_3) O_m^{(2)}(x_1 + x_2 + x_3) O_{ni}^{(4)}(x_1, x_2, x_3). \quad (3.20)$$

These algebraic equations define a matrix eigenvalue problem that is readily solved, once the individual overlap integrals have been done to compute the matrix elements. The results for a series of truncations at a fixed coupling of  $\lambda = 4\pi\mu^2$  are shown in Fig. 1.

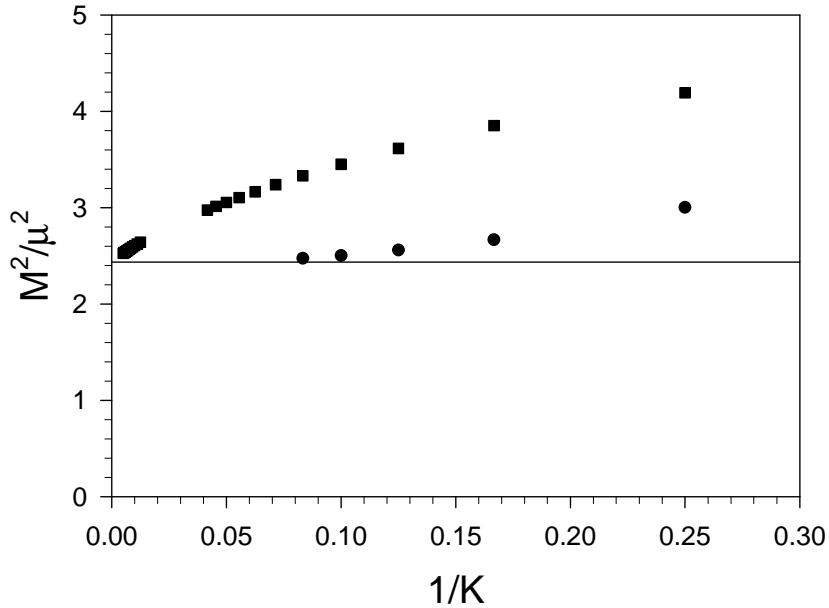


FIG. 1. Comparison of convergence rates for the fully symmetric polynomial basis (filled circles) and DLCQ (filled squares). The dimensionless eigenmass  $M^2/\mu^2$  is plotted versus  $1/K$ , the reciprocal of the basis order and of the DLCQ resolution, for the case where the coupling is  $\lambda = 4\pi\mu^2$ . The lower resolution DLCQ results cover a range up to  $K = 24$ ; the higher resolution results, a range of 80 to 200. The higher resolution points are used for an extrapolation to  $K = \infty$ , and the horizontal line is at the value of that limit.

For comparison, we consider a DLCQ approximation [8], which is, in the context of these integral equations, equivalent to a trapezoidal approximation to the integrations with a step size that is a reciprocal of an integer  $K$  and where the endpoint contributions are assumed to be zero. In DLCQ,  $K$  is called the harmonic resolution [8] or just ‘resolution.’ The individual momentum fractions are resolved into multiples of  $1/K$ . The neglect of the endpoints is the exclusion of zero modes, modes of zero longitudinal momentum. This is a



standard approximation in DLCQ, but it does delay convergence, because the wave functions go to zero slowly enough that the integrations do have endpoint contributions. Thus the neglect of zero modes induces errors of order  $1/K$ ; however, inclusion of them within the full many-body DLCQ approach is nontrivial [9].

The DLCQ approximation can be applied to either the coupled set or, equivalently, to the reduced two-boson equation, (3.12). We again obtain a matrix eigenvalue problem:

$$\frac{M^2}{\mu^2} \psi_{2m} = \left( \frac{K}{m} + \frac{K}{K-m} \right) \psi_{2m} + \frac{1}{2} \frac{\lambda}{4\pi\mu^2} \tilde{B}_{mn} \psi_{2n} + \frac{1}{2} \left( \frac{\lambda}{4\pi\mu^2} \right)^2 \tilde{C}_{mn} \psi_{2n} + \frac{1}{6} \left( \frac{\lambda}{4\pi\mu^2} \right)^2 \tilde{\Delta}_m \psi_{2m}, \quad (3.21)$$

where  $\psi_{2m} \equiv \psi_2(m/K, (K-m)/K)$  and

$$\tilde{B}_{mn} = \frac{K}{\sqrt{m(K-m)n(K-n)}}, \quad (3.22)$$

$$\tilde{C}_{mn} = \frac{1}{K^2} \sum_{n_1=1}^{m-n-1} \frac{1}{\frac{M^2}{\mu^2} - \frac{K}{n_1} - \frac{K}{m-n_1-n} - \frac{K}{n} - \frac{K}{K-m}} \frac{1}{\frac{n_1}{K} \frac{m-n_1-n}{K} \sqrt{\frac{m}{K} \frac{K-m}{K} \frac{n}{K} \frac{K-n}{K}}} \quad (3.23)$$

$$+ (m \leftrightarrow K-m, n \leftrightarrow K-n),$$

$$\tilde{\Delta}_m = \frac{1}{K^2} \sum_{n_1=1}^{m-1} \sum_{n_2=1}^{m-n_1-1} \frac{1}{\frac{M^2}{\mu^2} - \frac{K}{n_1} - \frac{K}{n_2} - \frac{K}{m-n_1-n_2} - \frac{K}{K-m}} \frac{1}{\frac{n_1}{K} \frac{n_2}{K} \frac{m-n_1-n_2}{K} \frac{m}{K}} + (m \leftrightarrow K-m). \quad (3.24)$$

However, the matrices themselves depend upon the eigenvalue. This (expected) complication for the reduced equation is easily managed, by iteration from a guess for the eigenmass. For a given value of  $M^2$  used in constructing the right-hand side, the lowest eigenvalue of the matrix can be computed and compared with the chosen value. If they do not agree, a new estimate of the eigenmass can be formed and the process repeated. We used the Muller algorithm [10] to guide the iterations; this improves on the more common secant algorithm with use of a quadratic, rather than linear, fit.

Some results are shown in Fig. 1. The low resolution DLCQ results are far from convergence. The high resolution results are quite close and easily extrapolated. However, these resolutions, from  $K = 80$  to  $200$ , are well beyond what can be used in practice for a many-body DLCQ calculation; the state of the art for  $\phi^4$  theory has been extended to  $K = 72$  on massively parallel machines [11]. The primary reason for DLCQ's slow convergence is the poor endpoint behavior. The expansions in terms of symmetric polynomials have the freedom to adjust the endpoint behavior in a very straightforward fashion. Convergence is then much more rapid.

#### IV. SUMMARY

We have derived an algorithm for the construction of fully symmetric orthonormal multivariate polynomials for the representation of the longitudinal momentum dependence of light-front wave functions for arbitrarily many bosons. The orthonormalization is carried out by the standard Gramm-Schmidt process. This process is applied to the symmetric polynomials for  $N$  bosons, obtained by considering all possible factorizations of the form  $\prod_{m=2}^N C_m^{n_m}$ , where the order  $n$  of the polynomial can be decomposed as  $n = \sum_m n_m$  and

$C_m$  is an order- $m$  polynomial with  $2 \leq m \leq N$ . The orthonormalization can be done relative to an inner product, such as (2.11), with a weight function chosen to optimize the utility of the polynomials. In particular, the weight function can be coordinated with the expected endpoint behavior of the wave functions to be represented. The example shown here, in Sec. III, illustrates the dramatically improved convergence, compared to the DLCQ method.

The original motivation in seeking these polynomials was in applications to equations obtained in the light-front coupled-cluster (LFCC) method [12, 13]. There the function of interest, to be expanded using these polynomials as a basis, is not a wave function but instead a vertex-like function that controls the operator that generates wave functions. This is done to avoid making a Fock-space truncation. However, the linearized version of the LFCC equations is equivalent to the Fock-space wave-function equations considered here.

The LFCC method will never use more than a small range of the boson multiplicity  $N$ ; for the current work on applications to  $\phi^4$  theory [13],  $N = 3$  and  $4$  are enough. Where the generalization to arbitrary  $N$  is important is in direct applications to Fock-state expansions for many-boson problems in light-front quantization. Specifically, the light-front many-body problem for  $\phi^4$  theory, which has been attempted only with DLCQ [11], can now be attacked with polynomial expansions in each Fock sector.

Such expansions are superior to DLCQ in two respects. One is the control of endpoint contributions, and the other is sector by sector control of resolution. The endpoint contributions are critical, not only for rapid convergence but also for computation of the vacuum expectation value for the  $\phi$  field when degenerate odd and even eigenstates are mixed.

The control of resolution in each sector is important for shifting computational resources to where they are most needed. In DLCQ, the number of discrete Fock states in each sector is fixed once the resolution is chosen, and this number is quite large for sectors with boson numbers near  $K/2$ , even though these sectors may not be particularly important for the calculation. With the polynomial expansion, the number and order of polynomials used in any Fock sector can be set individually, to place higher resolution in the sectors found to be most important for a given calculation.

## ACKNOWLEDGMENTS

This work was supported in part by the Minnesota Supercomputing Institute.

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